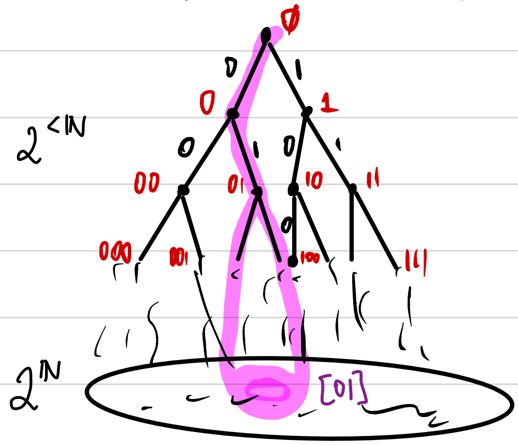
Math 451: Introduction to General Topology
Lecture 5

(g) Continued... It is not hard to show that d is a metric on $\Sigma^{\mathbb{N}}$, in fact, an ultrametric, i.e. the following strengthening of the Δ inequality holds:

$$d(x,z) \le \max\{d(x,y), d(y,z)\}.$$

Left as HW.



For $2^{-n} < r \le 2^{-(n+1)}$ and any $x \in \Sigma^{\mathbb{N}}$,

$$\begin{aligned} B_r(x) &= \{y \in \Sigma^{\mathbb{N}} : d(x,y) < r\} \\ &= \{y \in \Sigma^{\mathbb{N}} : d(x,y) \le 2^{-n}\} = \overline{B}_{2^{-n}}(x) \\ &= \{y \in \Sigma^{\mathbb{N}} : y|_n = x|_n\} =: [x|_n], \end{aligned}$$

where for any finite word $w \in \Sigma^{<\mathbb{N}}$, we define the cylinder at w to be

$$[w] := [w]_\Sigma := \{y \in \Sigma^{\mathbb{N}} : y|_{lh(w)} = w\},$$

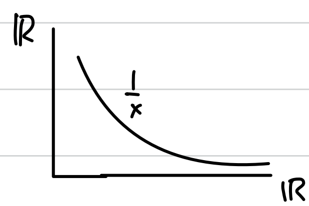
where $lh(w)$ denotes the length of w.

Observe that $\forall y \in B_r(x)$, $B_r(y) = [y|_n] = [x|_n] = B_r(x)$, so every element of a ball is a center of that ball.

We call $2^{\mathbb{N}}$ the Cantor space and $\mathbb{N}^{\mathbb{N}}$ the Baire space.

Def. For a metric space X, sets $Y, Z \subseteq X$, a point $x \in X$, define

- $\operatorname{diam}(Y) := \sup_{y_1, y_2 \in Y} d(y_1, y_2)$. Example: $\operatorname{diam}(0,1) = 1$.
- $d(x, Y) := \inf_{y \in Y} d(x,y)$.
- $d(Y, Z) := \inf_{y \in Y, z \in Z} d(y,z) = \inf_{y \in Y} d(y, Z)$. Example: $d(\operatorname{graph}(x \mapsto \frac{1}{x}), \mathbb{R}) = 0$.



Def. For metric spaces (X, d_X), (Y, d_Y), call a function $f: X \to Y$ an isometry if it preserves the metric, i.e. $d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2)$ for all $x_1, x_2 \in X$.

(Note that an isometry is always injective.) Call f an isometric isomorphism if it is a surjective isometry (hence it is bijective and both f, f^{-1} are isometries).

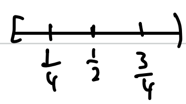
Def. For a metric space (X, d) and a subset $Y \subseteq X$, the restriction of d to Y (formally, restrict d to the set $Y \times Y$) is a metric on Y, which we will still denote by d. We say that (Y, d) is a subspace of (X, d).

Obs. Every isometry $f: X \to Y$ is an isometric isomorphism from X to the subspace $f(X)$.

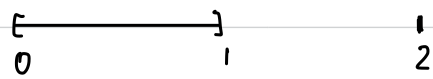
Prop. Let (X, d) be a metric space and $Y \subseteq X$. Then the balls in the subspace (Y, d) are exactly the intersections of balls of (X, d) with Y, i.e. $\forall y \in Y$ and $r \geq 0$,
$$B_r^Y(y) = B_r^X(y) \cap Y.$$

Proof. $B_r^Y(y) = \{x \in Y : d(y,x) < r\} = \{x \in X : d(y,x) < r\} \cap Y = B_r^X(y) \cap Y.$ $\boxed{\text{QED}}$

Examples. (a) In $Y = [0,1) \subseteq \mathbb{R}$,
$$B_2^Y(\tfrac{1}{2}) = [0,1).$$
$$B_1^Y(\tfrac{3}{4}) = [0,1).$$
$$B_{1/2}^Y(\tfrac{3}{4}) = (\tfrac{1}{4}, 1).$$
$$\overline{B}_{\frac{1}{2}}(\tfrac{1}{2}) = [0,1)$$



(b) In $Y := [0,1] \cup \{2\}$,
$$B_1(1) = (0,1].$$
$$\overline{B}_1(1) = [0,1] \cup \{2\} = Y.$$



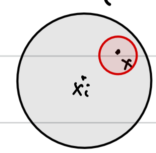
Caution. The closure of $B_1(1)$ is $\subseteq$ in the closed ball $\overline{B}_1(1)$, but isn't equal to it.

Open sets.

Def. In a metric space (X, d), a set $U \subseteq X$ is called open if it is an (arbitrary) union of (open) balls. Call a set $D \subseteq X$ closed if its complement $X \setminus D$ is open.

Prop. In a metric space (X,d), a set $U \subseteq X$ is open $\iff$ $\forall u \in U\ \exists r>0$ s.t. $B_r(u) \subseteq U$.

Proof. $\Rightarrow$. Let $U = \bigcup_{i \in I} B_i$ where each B_i is an open ball. Fix $x \in U$, so $x \in B_i = B_{r_i}(x_i)$.
Then $r := r_i - d(x_i, x) > 0$ and $B_r(x) \subseteq B_i(x_i) \subseteq U$.

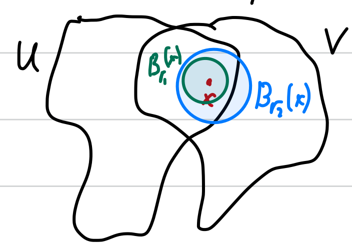


$\Leftarrow$. For every $u \in U$ $\exists\, r_u > 0$ (AC) s.t. $B_{r_u}(u) \subseteq U$. Then $U = \bigcup_{u \in U} B_{r_u}(u)$.

QED

Closure properties. In a metric space (X,d), the collection of open sets is closed under arbitrary unions and finite intersections. Consequently, closed sets are closed under arbitrary intersections and finite unions.

Proof. The closure of open sets under arbitrary unions is by definition. For finite intersections, it suffices to prove for the intersection of two open sets, by induction. Let $U, V \subseteq X$ be open. Fix $x \in U \cap V$. Then $x \in U$ hence $\exists\, B_{r_1}(x) \subseteq U$, $r_1 > 0$, and $x \in V$, so $B_{r_2}(x) \subseteq V$, $r_2 > 0$. Then $r := \min(r_1, r_2)$ works, i.e. $B_r(x) \subseteq B_{r_1}(x) \cap B_{r_2}(x) \subseteq U \cap V$.



QED

Examples.

(a) In $\mathbb{R}$ with the usual metric, open balls are bounded open intervals (a,b). Hence open sets are arbitrary unions of intervals. But in fact:

Prop. Every open subset of $\mathbb{R}$ is a ctbl union of pairwise disjoint intervals.

Proof. HW